Mathematics: The Core Course for A-level; L. Bostock and S. Chandler

Chapter 13 : Complex Numbers

Miscellaneous Ex 13

(1) "Show $|z_1 + z_2| \le |z_1| + |z_2|$" : Already done on P553.

$|z_1| = 6$ & $|z_2| = \sqrt{16+9} = 5$

$\therefore$ greatest value of $|z_1 + z_2|$ is $|z_1| + |z_2| = 6 + 5 = 11$.

The least value $|z_1| \sim |z_2| = \big|\,|z_1| - |z_2|\,\big|$

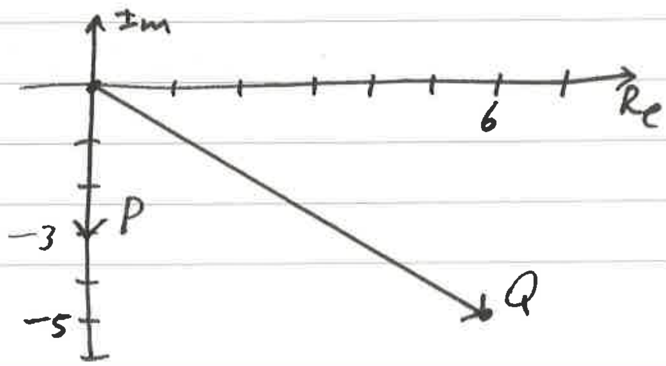
$= |6 - 5| = 1$

(2) $z_1 = \dfrac{2-i}{2+i} = \dfrac{2-i}{2+i} \cdot \dfrac{2-i}{2-i} = \dfrac{3-4i}{5} = \dfrac{3}{5} - \dfrac{4}{5}i$

$z_2 = \dfrac{2i-1}{1-i} = \dfrac{-1+2i}{1-i} \cdot \dfrac{1+i}{1+i} = \dfrac{-3+i}{2} = -\dfrac{3}{2} + \dfrac{i}{2}$

$P = 5z_1 + 2z_2 = (3-4i) + (-3+i) = -3i$

$Q = 5z_1 - 2z_2 = (3-4i) - (-3+i) = 6 - 5i$



(3) $z = -1 + 3i$

So $\quad z + \dfrac{2}{z} = -1 + 3i + \dfrac{2}{-1+3i}$

$= -1 + 3i + \dfrac{2}{-1+3i} \cdot \dfrac{-1-3i}{-1-3i}$

$= -1 + 3i + \dfrac{-2-6i}{8}$

$= -\dfrac{5}{4} + \dfrac{9}{4} i$

(4) (a) $z = 4 - 3i$

So $\quad z + \dfrac{1}{z} = 4 - 3i + \dfrac{1}{4-3i}$

$= 4 - 3i + \dfrac{1}{4-3i} \cdot \dfrac{4+3i}{4+3i}$

$= 4 + 3i + \dfrac{4+3i}{25}$

$= \dfrac{104}{25} + \dfrac{78}{25} i$ $\quad$ (Book Ans is wrong)

(b) $z = 4i \qquad \sqrt{z} = \sqrt{4i} = 2\sqrt{i}$

let $\sqrt{i} = a + bi$, so $i = (a+bi)^2 = (a^2 - b^2) + 2abi$

$\therefore$ Compare Coeffs: $\quad a^2 - b^2 = 0 \quad \& \quad 2ab = 1 \Rightarrow b = \dfrac{1}{2a}$

$\therefore \quad a^2 - \left(\dfrac{1}{2a}\right)^2 = 0 \Rightarrow 4a^4 - 1 = 0$

$a^4 = \dfrac{1}{4}$

$a^2 = \pm \dfrac{1}{2}$;

a is Real so $a^2 = -\frac{1}{2}$ is not valid

$\therefore \quad a^2 = \frac{1}{2} \Rightarrow a = \pm \frac{1}{\sqrt{2}}$

$\Rightarrow b = \pm \frac{\sqrt{2}}{2}$

So $\sqrt{z} = 2\sqrt{i} = 2\left(\pm \frac{\sqrt{2}}{2} i \pm \frac{1}{\sqrt{2}}\right)$

$= \pm 2\left(\frac{\sqrt{2}}{2} i + \frac{\sqrt{2}}{2}\right)$

$= \pm\sqrt{2}\,(1+i)$

④ $z_1 = 5-5i \; ; \; z_2 = -1+7i$

$|z_1 + z_2| = \sqrt{(5-1)^2 + (-5+7)^2} = \sqrt{20}$

$|z_1 - z_2| = \sqrt{(5+1)^2 + (-5-7)^2} = \sqrt{180}$

$|z_1| + |z_2| = \sqrt{25+25} + \sqrt{1+49} = 2\sqrt{50} = \sqrt{200}$

$\therefore \quad \sqrt{20} < \sqrt{180} < \sqrt{200}$

⑤ $z = \frac{5+12i}{3+4i} = \frac{5+12i}{3+4i} \cdot \frac{3-4i}{3-4i}$

$= \frac{(15+48) + (-20+36)i}{25}$

$= \frac{63}{25} + \frac{16}{25} i$

$\theta = \tan^{-1}\left(\frac{16/25}{63/25}\right) = 14.25^\circ = 0.249 \text{ Rad}$

$r = \sqrt{\left(\frac{63}{25}\right)^2 + \left(\frac{16}{25}\right)^2} = \frac{13}{5}$

So $Z = \frac{13}{5}\left(\cos 0.249 + i \sin 0.249\right)$ (in Rad)

Since, in $\tan^{-1}\frac{y}{x}$, y is The $\sin\theta$ term

& x " " $\cos\theta$ "

then $\cos\theta = \frac{63}{25}$

&

$\sin\theta = \frac{16}{25}$

(6) $Z_1 = \dfrac{a}{1+i}$, $Z_2 = \dfrac{b}{1+2i}$ & $Z_1 + Z_2 = 1$

So $\dfrac{a}{1+i} + \dfrac{b}{1+2i} = \dfrac{a(1+2i) + b(1+i)}{-1+3i} = 1$

$\therefore (a+b) + i(2a+b) = -1 + 3i$

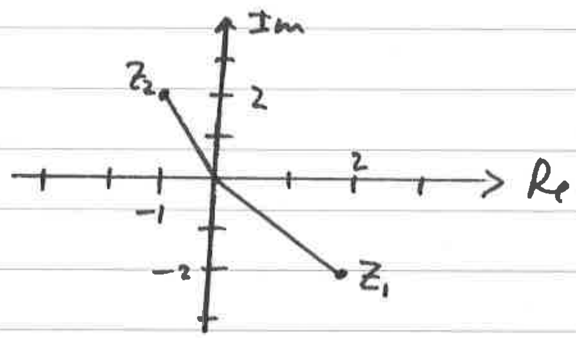
$$\therefore \left.\begin{aligned} a + b &= -1 \\ 2a + b &= 3 \end{aligned}\right\} \text{ subtract: } -a = -4,\ a = 4$$

$\Rightarrow b = -5$

So $Z_1 = \dfrac{4}{1+i}$, $Z_2 = \dfrac{-5}{1+2i}$

$Z_1 = \dfrac{4}{1+i} \cdot \dfrac{1-i}{1-i}$, $Z_2 = -\dfrac{5}{1+2i} \cdot \dfrac{1-2i}{1-2i}$

$= 2 - 2i$ $\qquad = -1 + 2i$



horizontal distance between Z_1 & Z_2 : $2-(-1) = 3$
vertical " " " : $2-(-2) = 4$

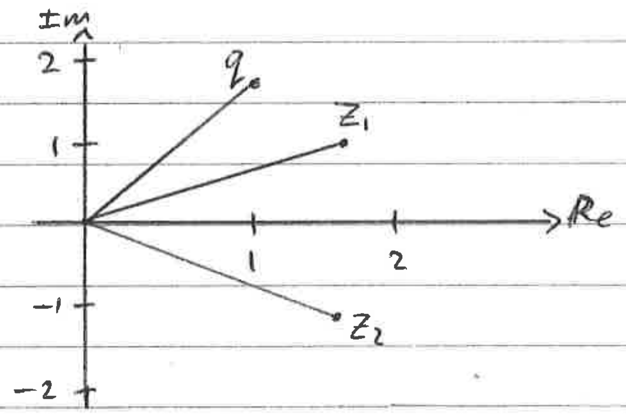
So $|Z_1 + Z_2| = \sqrt{9+16} = 5$

(7) $Z_1 = \sqrt{3} + i$ So $|Z_1| = \sqrt{3+1} = 2$

$\arg(Z_1) = \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$

$$q = \frac{Z_1}{Z_2} = \frac{\sqrt{3}+i}{\sqrt{3}-i} \cdot \frac{\sqrt{3}+i}{\sqrt{3}+i}$$

$$= \frac{2+2\sqrt{3}i}{4} = \frac{1}{2} + \frac{\sqrt{3}}{2} i$$



$Z_1 - Z_2 = (\sqrt{3}+i) - (\sqrt{3}-i) = 2i$
$q - Z_1 = (1+\sqrt{3}i) - (\sqrt{3}+i) = (1-\sqrt{3}) - (\sqrt{3}+1)i$

$|Z_1 - Z_2| = 2$, $|q - Z_1| = \sqrt{(1-\sqrt{3})^2 + (\sqrt{3}+1)^2} = \sqrt{8}$

Note that $2 \neq \sqrt{8}$; In fact, I believe the question has been wrongly written. They may have been trying to write a locus question (where we could ask for $|Z - Z_1| = |q - Z|$) but I am just guessing.

(8) $(1+5i)p - 2q = 3 + 7i$

(a) $p, q \in \mathbb{R}$

$$p + 5pi - 2q = 3 + 7i$$

Re : $p - 2q = 3$

Im : $5p = 7 \longrightarrow p = \frac{7}{5}$

$$\therefore q = \frac{p-3}{2} = -\frac{4}{5}$$

(b) p, q are complex conjugates

let $p = a + ib$ & $q = a - ib$

$\therefore$ we have $(1+5i)(a+ib) - 2(a-ib) = 3 + 7i$

So $a + i(b+5a) + 5bi^2 - 2a + 2bi = 3 + 7i$

Re : $-a - 5b = 3$

Im : $3b + 5a = 7$ $\Big\}$ Solve : $a = 2$, $b = -1$

$\therefore$ $p = 2 - i$, $q = 2 + i$.

(9) $z_1 = 1 - i$, $z_2 = 7 + i$

(a) $z_1 - z_2 = -6 - 2i$

So $|z_1 - z_2| = \sqrt{36 + 4} = \sqrt{40} = 2\sqrt{10}$

(b) $z_1 z_2 = (1-i)(7+i) = 1 + i - 7i - i^2$

$$= 2 - 6i$$

$|z_1 z_2| = \sqrt{4 + 36} = \sqrt{40} = 2\sqrt{10}$

(c) $\dfrac{z_1 - z_2}{z_1 z_2} = \dfrac{-6 - 2i}{2 - 6i} \cdot \dfrac{2 + 6i}{2 + 6i}$

$$= \frac{-40i}{40} = -i$$

$\left|\dfrac{z_1 - z_2}{z_1 z_2}\right| = 1$, which happens to be the same as if we had done

$$\frac{|z_1 - z_2|}{|z_1 z_2|} = \frac{2\sqrt{10}}{2\sqrt{10}} = 1 \quad (!)$$

(This is not a coincidence but is a general property of the mod of complex N°s)

(10) $a = 3 - i \quad , \quad b = 1 + 2i$

(a) $2a + 3b = 6 - 2i + 3 + 6i$

$= 9 + 4i$

$|2a + 3b| = \sqrt{81 + 16} = \sqrt{97}$

(b) $\frac{a}{2b} = \frac{3-i}{2+4i} = \frac{3-i}{2+4i} \cdot \frac{2-4i}{2-4i}$

$= \frac{2 - 14i}{20} = \frac{1}{10} - \frac{7}{10} i$

$\left|\frac{a}{2b}\right| = \sqrt{\frac{1}{100} + \frac{49}{100}} = \sqrt{\frac{50}{100}} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$

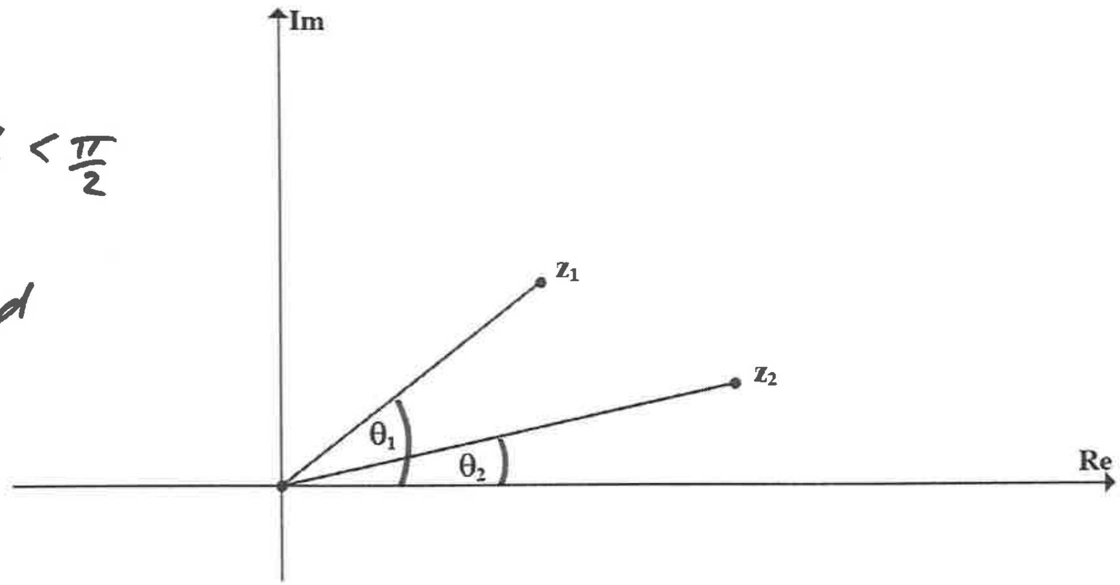
(11)

(11) Let $z_1 = a + ib$

$z_2 = c + id$

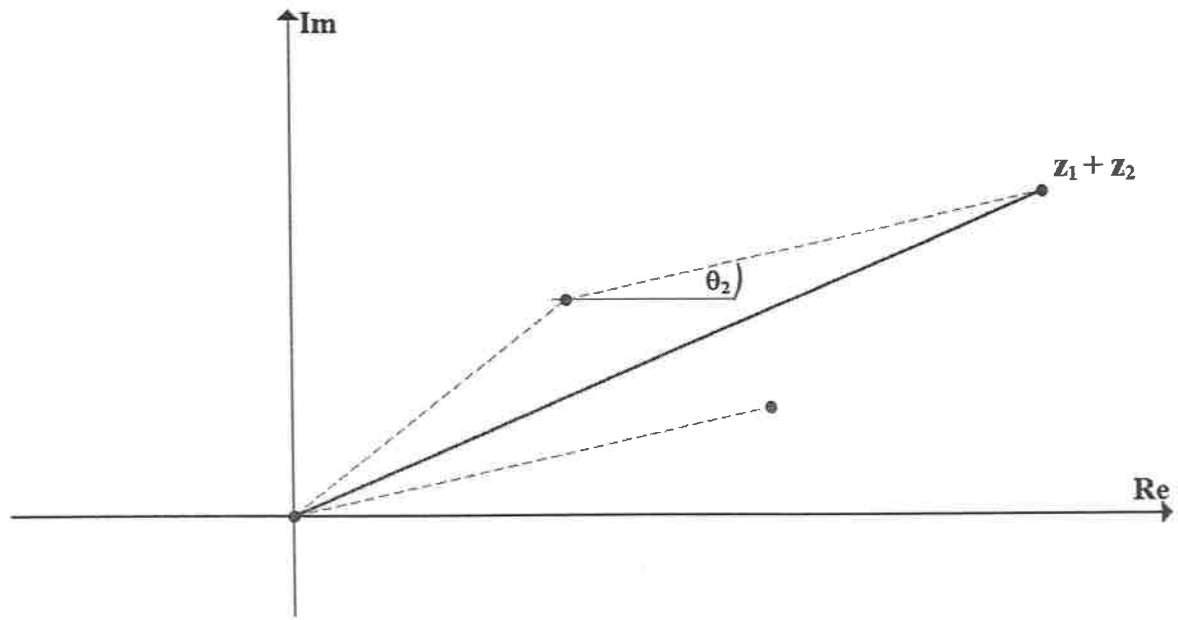
as shown, where $0 < \arg z_2 < \arg z_1 < \frac{\pi}{2}$

It does not matter what values a, b, c, d are.



$z_1 + z_2$ is formed by starting at z_1 and moving up a distance of z_2 at an angle θ_2 to the horizontal of z_1

$z_1 + z_2$ will be higher than z_1 since we are adding an angle θ_2 from our starting position z_1



Homework

1. Find the inverse of the matrix $A = \begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}$.

Hence solve the system of equations

$$\begin{aligned} 3x + 2y &= 4 \\ -2x + y &= 6 \end{aligned}$$

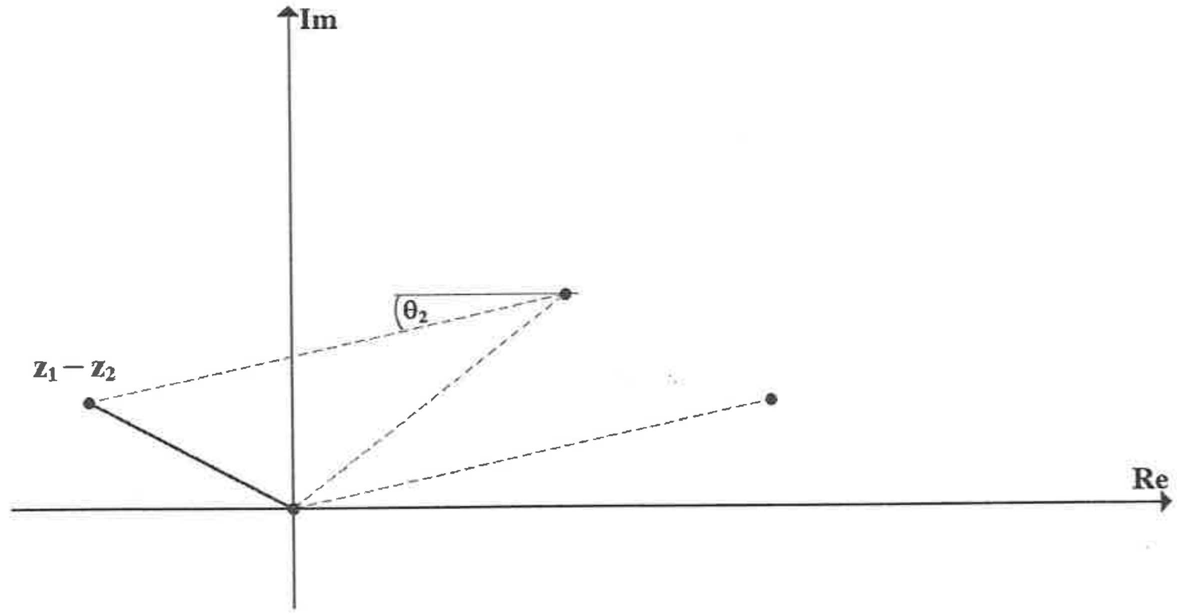
2. Find the inverse of the matrix $A = \begin{pmatrix} 2 & 2 & 1 \\ -1 & 1 & -1 \\ 3 & 1 & 3 \end{pmatrix}$ and hence solve the system of equations

$$\begin{aligned} 2x + 2y + z &= 4 \\ -x + y - z &= 1 \\ 3x + y + 3z &= 1 \end{aligned}$$

3. Solve the system of equations

$$\begin{aligned} x + 2y - z &= 8 \\ 6x - y - 3z &= -4 \\ 3x + y + 2z &= 4 \end{aligned}$$

$z_1 - z_2$ is formed by starting at z_1 & moving down a distance z_2, This "down" being given by angle θ_2 to the horizontal of z_1



5. If $A = \begin{pmatrix} -1 & 3 & 4 \\ 4 & 4 & 0 \\ 2 & 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 3 & 4 \\ 4 & 6 & 4 \\ 4 & 4 & 0 \end{pmatrix}$, find a matrix Q such that QA = B.

(12) $z = \cos\theta + i.\sin\theta$

$$1 + z = (1+\cos\theta) + i.\sin\theta$$

$$\frac{1}{1+z} = \frac{1}{(1+\cos\theta) + i.\sin\theta}$$

$$= \frac{1}{(1+\cos\theta) + i.\sin\theta} \cdot \frac{(1+\cos\theta) - i.\sin\theta}{(1+\cos\theta) - i.\sin\theta}$$

$$= \frac{(1+\cos\theta) - i.\sin\theta}{(1+\cos\theta)^2 + \sin^2\theta}$$

$$= \frac{(1+\cos\theta) - i.\sin\theta}{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \frac{(1+\cos\theta) - i.\sin\theta}{2 + 2\cos\theta}$$

$$= \frac{1+\cos\theta}{2+2\cos\theta} - \frac{1}{2}i\,\frac{\sin\theta}{1+\cos\theta}$$

At This point, always Remember your trig identities in order to change or simplify Things (This also applies when doing integration Exercises)

So $\dfrac{\sin\theta}{1+\cos\theta} = \dfrac{\sin 2(\theta/2)}{1+\cos 2(\theta/2)}$.

This is a trick often done so as to convert to half-angle or t-formulae.

$$\therefore \quad \frac{\sin\theta}{1+\cos\theta} = \frac{\sin 2(\theta/2)}{1+\left[2\cos^2\left(\frac{\theta}{2}\right)-1\right]}$$

$$= \frac{2\sin(\theta/2)\cdot\cos(\theta/2)}{2\cos^2\left(\frac{\theta}{2}\right)}$$

$$= \frac{\sin\theta/2}{\cos\theta/2} = \tan\theta/2$$

Hence

$$\frac{1}{1+z} = \frac{1}{2} - \frac{1}{2}i\tan\frac{\theta}{2} \quad ✓$$

(a) $2z = 2\cos\theta + 2i\sin\theta$

$$z^2 = \cos^2\theta + 2i\cos\theta\sin\theta + i^2\sin^2\theta$$

$$= \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta$$

So

$$\frac{2z}{1+z^2} = \frac{2\cos\theta + 2i\sin\theta}{1+\cos^2\theta-\sin^2\theta+2i\cos\theta\sin\theta}.$$

To make it Easier to Read let $\cos\theta = c$
$\sin\theta = s$

$$\therefore \quad \frac{2z}{1+z^2} = \frac{2c+2is}{1+c^2-s^2+2ics}\cdot\frac{1+c^2-s^2-2ics}{1+c^2-s^2-2ics}$$

To make it easier to multiply out, I will look at The RHS of above as

$$\frac{2A+2Bi}{C+2iD},$$ & Expand in Blocks of A, B, C, D.

So

$$\frac{2z}{1+z^2} = \frac{2c(1+c^2-s^2)+4cs^2+i\left[2s(1+c^2-s^2)-4c^2s\right]}{(1+c^2-s^2)^2+4(cs)^2}$$

Now Expand & Simplify

$$\frac{2z}{1+z^2} = \frac{2c\ (1+c^2+s^2)\ +i\left[2s\ (1-c^2-s^2)\right]}{s^4+(2c^2-2)s^2+c^4+2c^2+1}$$

Factorise denominator & Remember $c^2+s^2=1$ (trig id.)

$$\frac{2z}{1+z^2} = \frac{4c\ +\ i(0)}{(s^2-2s+c^2+1)(s^2+2s+c^2+1)}$$

$$= \frac{4c}{(2-2s)(2+2s)}$$

$$= \frac{\cos\theta}{(1-\sin\theta)(1+\sin\theta)}$$

$$= \frac{\cos\theta}{1+\sin\theta-\sin\theta-\sin^2\theta}$$

$$= \frac{\cos\theta}{\cos^2\theta} = \sec\theta.$$

(b) $$\frac{1-z^2}{1+z^2} = \frac{1-[c^2-s^2+2ics]}{1+[c^2-s^2+2ics]}$$

$$= \frac{1-c^2+s^2-2ics}{1+c^2-s^2+2ics} \cdot \frac{1+c^2-s^2-2ics}{1+c^2-s^2-2ics}$$

$$= \frac{(1-c^2+s^2)(1+c^2-s^2)-4(cs)^2}{s^4+(2c^2-2)s^2+c^4+2c^2+1}$$

$$-i\left[\frac{2cs(1-c^2+s^2)+2cs(1+c^2-s^2)}{s^4+(2c^2-2)s^2+c^4+2c^2+1}\right]$$

--- too long & complicated so try something else, namely long division:

$$\begin{array}{r} -1 \\ -z^2+1 \,\big)\, \overline{z^2+1} \\ \underline{z^2-1} \\ 2 \end{array}$$

$$\Rightarrow \text{so } \frac{1-z^2}{1+z^2} = -1 + \frac{2}{1+z^2}.$$

Hence, since $z^2 = c^2-s^2+2ics$:

$$\frac{1-z^2}{1+z^2} = -1+\frac{2}{1+z^2} = -1+\frac{2}{1+c^2-s^2+2ics}$$

$$= -1+\frac{2}{1+c^2-s^2+2ics} \cdot \frac{1+c^2-s^2-2ics}{1+c^2-s^2-2ics}$$

$$= -1+\frac{2(1+c^2-s^2-2ics)}{(1+c^2-s^2)^2+4(cs)^2}$$

$$\frac{1-z^2}{1+z^2} = -1 + \frac{2(1+c^2-s^2-2ics)}{1+2(c^2-s^2)+(c^2-s^2)^2+4c^2s^2}$$

$$= -1 + \frac{2(1+c^2-s^2-2ics)}{1+2c^2-2s^2+c^4-2c^2s^2+s^4+4c^2s^2}$$

$$= -1 + \frac{2(1+c^2-s^2)-4ics}{s^4+s^2(2c^2-2)+c^4+2c^2+1}$$

$$= -1 + \frac{2(1+c^2-s^2)-4ics}{s^4+2s^2(c^2-1)+(c^2+1)^2}$$

In The Numerator $1-s^2=c^2$ by Trig identity. Then, by also factorizing The denominator we get:

$$\frac{1-z^2}{1+z^2} = -1 + \frac{4c^2-4ics}{(s^2-2s+c^2+1)(s^2+2s+c^2+1)}$$

Again using $s^2+c^2=1$ The Denom simplifies to

$$\frac{1-z^2}{1+z^2} = -1 + \frac{4c^2-4ics}{(2-2s)(2+2s)}$$

$$= -1 + \frac{c^2-ics}{(1-s)(1+s)}$$

$$= -1 + \frac{\cos^2\theta}{1-\sin^2\theta} - i\frac{\cos\theta\sin\theta}{1-\sin^2\theta}$$

$$= -1 + 1 - i\frac{\cos\theta\sin\theta}{\cos^2\theta} = -i\tan\theta.$$

Still quite long to solve but relatively more manageable.

As an Extra Ex (Not in The book) show That

i) $$\frac{1-z}{1+z} = -i \tan\frac{\theta}{2}$$

ii) $$\frac{1}{1-z^2} = \frac{1}{2}\left[1 + i\left(\tan\frac{\theta}{2} + \cos\frac{\theta}{2}\right)\right]$$

(for ii) use partial fractions, then later ~~on~~ use a suitable identity for $\cos 2\left(\frac{\theta}{2}\right)$)

Here is an Extra which is Not part of The question:

"Show That $\dfrac{2z}{1+z} = 1 + i.\tan\left(\frac{\theta}{2}\right)$"

<u>Sol</u>

$$\frac{2z}{1+z} = 2(\cos\theta + i\cdot\sin\theta)\cdot\frac{1}{2}\left(1 - i\tan\left(\frac{\theta}{2}\right)\right)$$

$$= \cos\theta + i\left(-\cos\theta \tan\frac{\theta}{2} + \sin\theta\right)$$

$$- i^2 \cdot \sin\theta \tan\frac{\theta}{2}$$

$$= \cos\theta + \sin\theta \tan\frac{\theta}{2} + i\left(\sin\theta - \cos\theta \tan\frac{\theta}{2}\right)$$

$$= \cos 2\left(\frac{\theta}{2}\right) + \sin 2\left(\frac{\theta}{2}\right) \tan\left(\frac{\theta}{2}\right)$$

$$+ i\left(\sin 2\left(\frac{\theta}{2}\right) - \cos 2\left(\frac{\theta}{2}\right) \tan\left(\frac{\theta}{2}\right)\right)$$

We do This so as to convert to half angles. Now use trig identities to Expand.

$$\frac{2z}{1+z} = \cos 2\left(\frac{\theta}{2}\right) + 2\sin\frac{\theta}{2}.\cos\frac{\theta}{2} \tan\frac{\theta}{2}$$

$$+ i.\left[\sin 2\left(\frac{\theta}{2}\right) - \left(2\cos^2\left(\frac{\theta}{2}\right) - 1\right) \tan\frac{\theta}{2}\right]$$

$$= \cos 2\left(\frac{\theta}{2}\right) + 2\sin^2\left(\frac{\theta}{2}\right)$$

$$+ i.\left(\sin 2\left(\frac{\theta}{2}\right) - 2\cos^2\left(\frac{\theta}{2}\right)\tan\frac{\theta}{2} + \tan\frac{\theta}{2}\right)$$

$$= \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) + 2\sin^2\left(\frac{\theta}{2}\right)$$

$$+ i.\left(2\sin\frac{\theta}{2}.\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \tan\frac{\theta}{2}\right)$$

$$= \cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) + i\tan\left(\frac{\theta}{2}\right)$$

$$= 1 + i.\tan\left(\frac{\theta}{2}\right) \checkmark$$

13 (a) $z = a + ib$, $\bar{z} = a - ib$

So $z.\bar{z} + 2iz = (a^2+b^2) + 2i(a+ib)$

$= (a^2+b^2) + 2bi^2 + 2ia$

$= 12 + 6i$

So $a^2+b^2-2b = 12$ and $2a = +6 \Rightarrow a = +3$

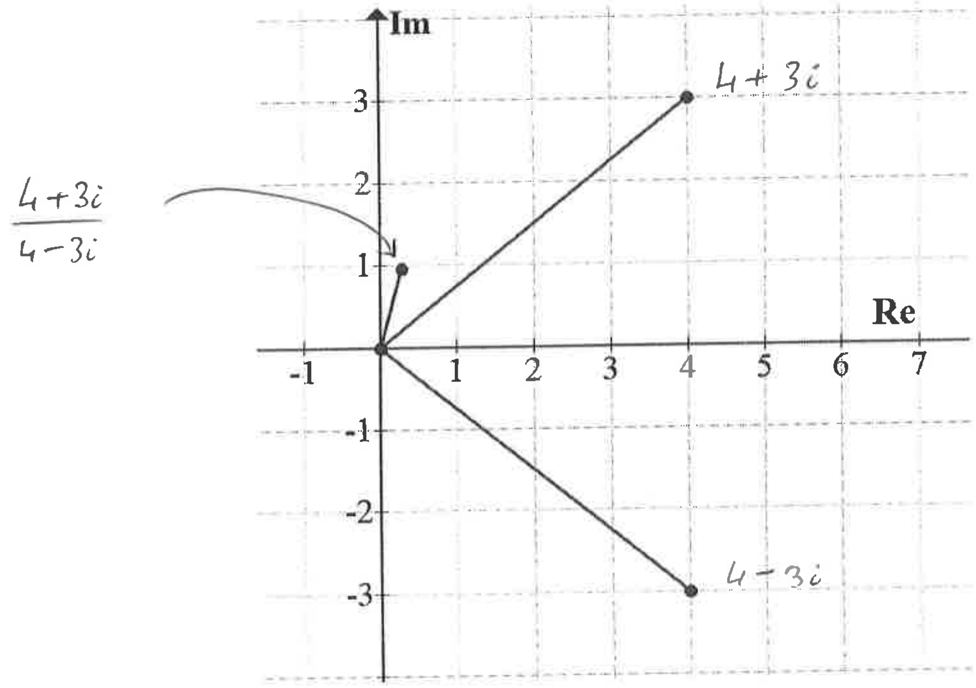
For $a = +3$: $9 + b^2 - 2b = 12$

So $b^2 - 2b - 3 = 0$

$\therefore (b-3)(b+1) = 0 \Rightarrow b = +3, -1$

So $z = 3 + 3i$ or $3 - i$

(b)



Calculation : $\frac{4+3i}{4-3i} \cdot \frac{4+3i}{4+3i} = \frac{16-9+24i}{16+9}$

$= \frac{7}{25} + \frac{24}{25} i$

Reading to Learn Mathematics

The End

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(14) (a) $z = 3+4i$

$$z + \frac{25}{z} = 3+4i + \frac{25}{3+4i}$$

$$= 3+4i + \frac{25}{3+4i} \cdot \frac{3-4i}{3-4i}$$

$$= 3+4i + \frac{75-100i}{25}$$

$$= 3+4i + 3-4i$$

$$= 3.$$

Note That 25 is The mod Squared of z. Because of This we get The Real part of z.

In general $z + \frac{|z|^2}{z} = Re(z) = a$ for a complex No $z = a+ib$.

<u>Proof</u> : $a+ib + \frac{|z|^2}{a+ib} = a+ib + \frac{|z|^2}{a+ib} \cdot \frac{a-ib}{a-ib}$

$$= a+ib + \frac{|z|^2(a-ib)}{a^2+b^2}$$

$$= a+ib + \frac{|z|^2}{|z|^2}(a-ib)$$

$$= a = Re(z)$$

Similarly $z - \frac{|z|^2}{z} = Im(z) = b$

(b) $z = x + iy$

So $z + \frac{1}{z} = x + iy + \frac{1}{x+iy}$

$= x + iy + \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy}$

$= x + iy + \frac{x-iy}{x^2+y^2}$

$= x + iy + \frac{1}{|z|^2}(x - iy)$

So $\mathrm{Re}\left(z + \frac{1}{z}\right) = x + \frac{x}{|z|^2}$

& $\mathrm{Im}\left(z + \frac{1}{z}\right) = y - \frac{y}{|z|^2}$

(15) (a) Let $z = 5 + 12i$

Then $\sqrt{z} = \sqrt{5+12i} = a + ib$

So $5 + 12i = (a+ib)^2$
$= a^2 + 2abi + i^2b^2$
$= a^2 - b^2 + 2ab\,i$

Compare Re & Im

Re : $5 = a^2 - b^2$

Im : $12 = 2ab \longrightarrow b = \frac{6}{a}$

So $5 = a^2 - \frac{36}{a^2} \implies a^4 - 5a^2 - 36 = 0$

So $(a^2 - 9)(a^2 + 4) = 0$

$\therefore \quad a^2 = 9$ or $a^2 = -4$

But a is Real so $a^2 = -4$ is Not valid, $\therefore a = \pm 3$

$\Rightarrow b = \pm 2$

So $\sqrt{5+12i} = \pm(3+2i)$

(b) by "Amplitude" They mean "arguement". "Amplitude" tends to mean size (ie. modulus) Not angle, so I think the question is badly worded.

i) $\underline{1-i}$: $\quad mod(1-i) = \sqrt{1+1} = \sqrt{2} = |z_1|$

$\theta_1 = \tan^{-1} \frac{-1}{1} = -\frac{\pi}{4}$

ii) $\underline{4+3i}$: $\quad mod(4+3i) = \sqrt{16+9} = 5 = |z_2|$

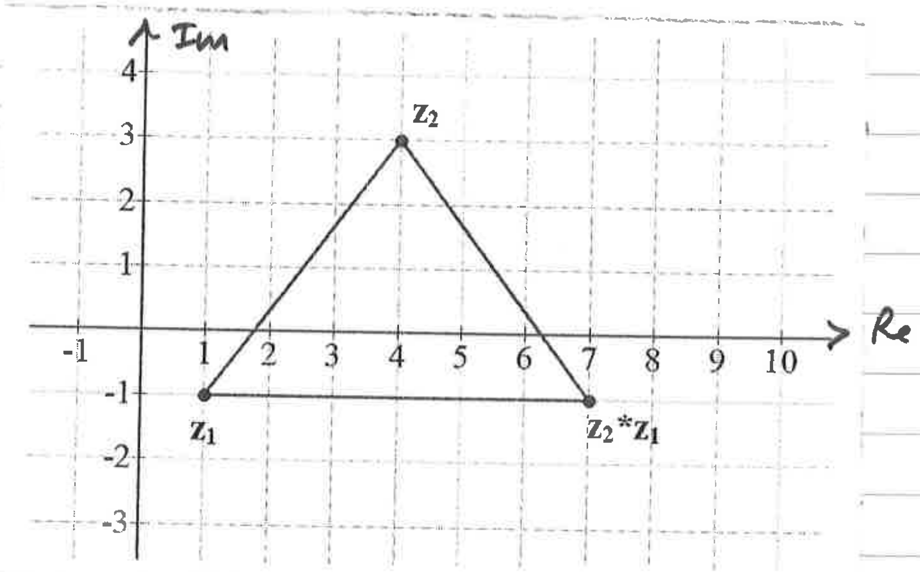
$\theta_2 = \tan^{-1} \frac{3}{4} = 0.6435$ Radians.

iii) by properties of Complex Numbers

$|z_1 . z_2| = |z_1|\,|z_2|$ & $\arg(z_1 z_2) = \theta_1 + \theta_2$

So $\arg[(1-i)(4+3i)] = -\frac{\pi}{4} + 0.6435 = -0.1412$ Radians

& $|(1-i)(4+3i)| = 5 \times \sqrt{2}$



$(1-i)(4+3i) = 7-i$

From the graph & the calculations we see that z_1 is in line with $z_1 \cdot z_2$.

This allows us to drop a perpendicular from z_2 to this line & use $A = \frac{1}{2} b h$ as our formula.

Length of base : $z_1 \to z_1 z_2$: 6 units

i.e $7-i-(1-i) = 6$

height : $z_2 \to$ line joining z_1 & $z_1 z_2$: 4 units

i.e $4+3i - \underbrace{(4-i)}_{} = 4$

point directly under z_2 on the line $z_1 \to z_1 z_2$

So $A = \frac{1}{2} \cdot 6 \cdot 4 = 12$ units^2

(16) $\dfrac{(i+1)^2}{(i-1)^4}$

(a)

$|z_1| = |i+1| = \sqrt{2}$, So $|z_1^2| = (\sqrt{2})^2 = 2$

$|z_2| = |i-1| = \sqrt{2}$, so $|z_2^4| = (\sqrt{2})^4 = 4$

$\theta_1 = \tan^{-1} \frac{1}{1} = \frac{\pi}{4}$, $\theta_2 = \tan^{-1} \frac{-1}{1} = \frac{-\pi}{4}$

$\arg(z_1^2) = 2\theta_1 = \frac{\pi}{2}$, $\arg(z_2^4) = 4\theta_2 = -\pi$

So $\dfrac{|z_1^2|}{|z_2^4|} = \dfrac{2}{4} = \frac{1}{2}$

$\dfrac{\arg(z_1^2)}{\arg(z_2^4)} = \arg(z_1^2) - \arg(z_2^4) = \frac{\pi}{2} - \pi = -\pi/2$

(b) $z = 5-12i$, $\sqrt{z} = \sqrt{5-12i} = a+ib$

So $5-12i = (a+ib)^2$

$= a^2 - b^2 + 2abi$

So $5 = a^2 - b^2$ & $-12 = 2ba$, so $b = -\frac{6}{a}$

$\therefore$ $5 = a^2 - \dfrac{36}{a^2}$ $\Rightarrow$ $a^4 - 5a^2 - 36 = 0$

This is a quadratic in a^2, so

$a^2 = \dfrac{5 \pm \sqrt{25+144}}{2} = 9, -4$

But a is Real so $a^2 = -4$ is Not Valid

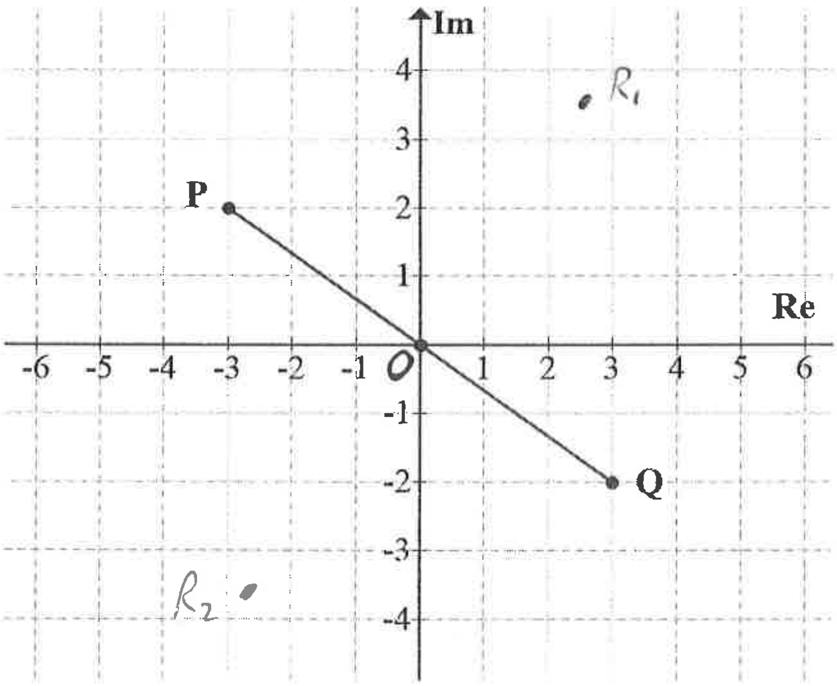
$$\therefore a^2 = 9 \Rightarrow a = \pm 3$$

$$\therefore b = \mp 2$$

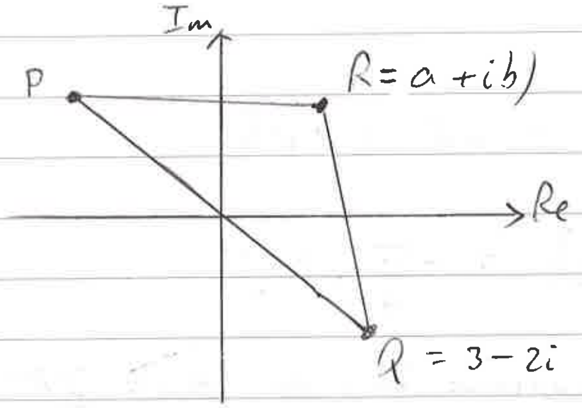
$$\therefore \sqrt{5 - 12i} = 3 - 2i, \; -3 + 2i$$

The line Joining P & Q is a Straight Line Through the origin, so R_1 & R_2 must lie on either Side of this line.

Also, The origin is The half Way point between P & Q so The vertex of The two Triangles will lie directly opposite The origin.



Using Basic coordinate geometry (using distance between two points) we have



$$|RQ| = \sqrt{(3-a)^2 + (2+b)^2} = 2\sqrt{13}$$

$$|RP| = \sqrt{(a+3) + (b-2)^2} = 2\sqrt{13}$$

Expand & solve for a and b:

$$(3-a)^2 + (2+b)^2 = 52$$

& $$(a+3)^2 + (b-2)^2 = 52$$

So $$9 - 6a + a^2 + 4 + 4b + b^2 = 52 \quad (1)$$

& $$a^2 + 6a + 9 + b^2 - 4b + 4 = 52 \quad (2)$$

(1) − (2) : $-12a + 8b = 0 \Rightarrow b = \frac{3}{2}a$ (3)

(1) + (2) : $$2a^2 + 18 + 2b^2 + 8 = 104$$

$$a^2 + 9 + b^2 + 4 = 52 \quad (4)$$

Sub (3) into (4) : $a^2 + 9 + \left(\frac{3}{2}a\right)^2 + 4 = 52$

Solve to get $a = \pm\sqrt{12} = \pm 2\sqrt{3}$

$\therefore\ b = \pm 3\sqrt{3}$

hence $R = \pm(2 + 3i)\sqrt{3}$

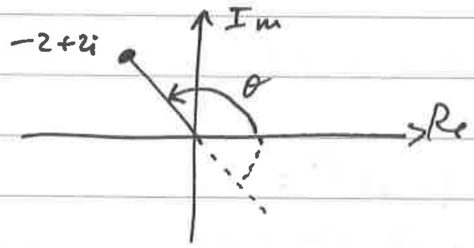
(17) (a) $z^2+4z+8=0$

$$z = \frac{-4 \pm \sqrt{16-32}}{2} = -2 \pm 2i$$

So $|-2+2i| = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$$\arg(-2+2i) = \tan^{-1}\frac{2}{-2} = -\frac{\pi}{4} = \theta$$

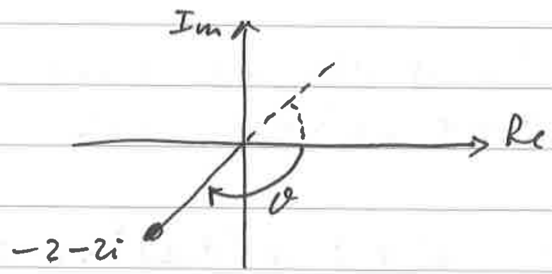
But $-2+2i$ lies in quadrant II, so $\theta = \frac{3\pi}{4}$



For $|-2-2i| = \sqrt{4+4} = 2\sqrt{2}$

$$\arg(-2-2i) = \tan^{-1}\frac{-2}{-2} = \frac{\pi}{4} = \theta$$

But $-2-2i$ lies in quadrant III so $\theta = -\frac{3\pi}{4}$



Now let $\alpha = -2+2i$ & $\beta = -2-2i$

$\alpha+\beta = -4$, $\alpha\beta = 8$.

So $$\frac{\alpha+\beta+4i}{\alpha\beta+8i} = \frac{-4+4i}{8+8i} = \frac{-1+i}{2+2i}$$

So $\dfrac{\alpha+\beta+4i}{\alpha\beta+8i} = \dfrac{-1+i}{2+2i} \cdot \dfrac{2-2i}{2-2i}$

$$= \frac{-2+2i+2i+2}{4+4} = \frac{1}{2}i.$$

(b) $z_1 = -1+i\sqrt{3}$, $z_2 = \sqrt{3}+i$

$z_1 + z_2 = (-1+\sqrt{3}) + i(1+\sqrt{3})$

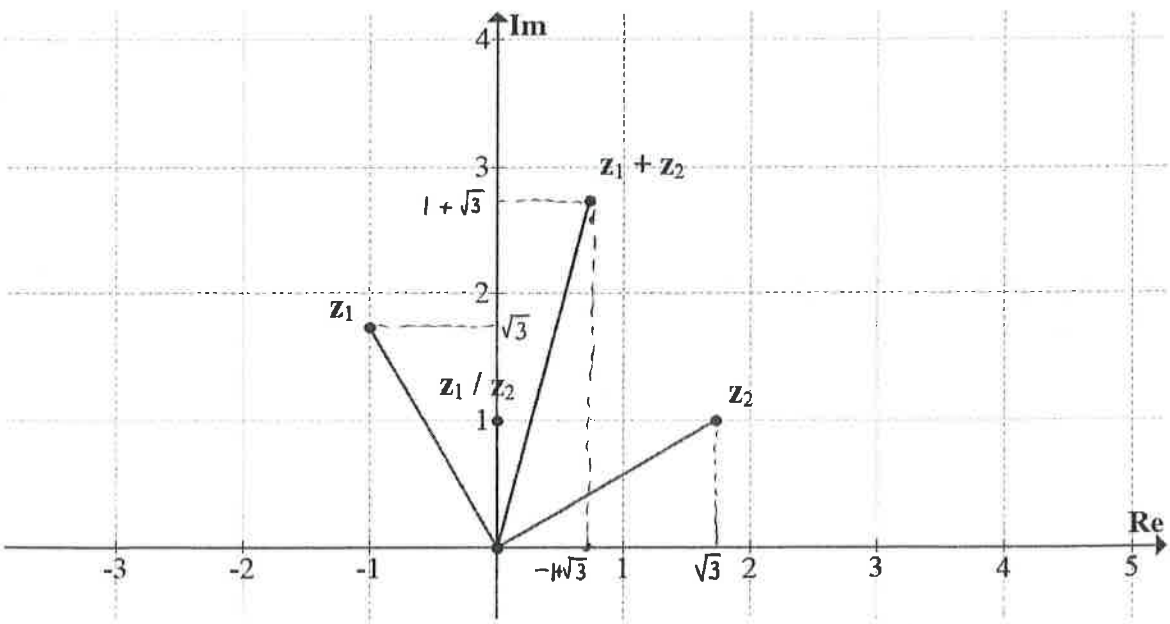
$z_1 \cdot z_2 = (-1+i\sqrt{3})(\sqrt{3}+i)$

$= -\sqrt{3} - i + 3i + i^2\sqrt{3}$

$= -2\sqrt{3} + 2i$

$\dfrac{z_1}{z_2} = \dfrac{-1+i\sqrt{3}}{\sqrt{3}+i} = \dfrac{-1+i\sqrt{3}}{\sqrt{3}+i} \cdot \dfrac{\sqrt{3}-i}{\sqrt{3}-i}$

$$= \frac{-\sqrt{3}+i+3i-i^2\sqrt{3}}{4} = i$$



(18) (a) i) $\dfrac{1-i}{(3-i)^2} = \dfrac{1-i}{8-6i}$

$$= \frac{1-i}{8-6i} \cdot \frac{8+6i}{8+6i}$$

$$= \frac{8+6i-8i-6i^2}{64-36i^2}$$

$$= \frac{14-2i}{100} = \frac{7}{50} - \frac{1}{50}i$$

ii) $(c+i)^4 = c^4 + 4c^3i + 6c^2i^2 + 4ci^3 + i^4$ (by Binomial)

$$= c^4 + 4c^3i - 6c^2 - 4ci + 1$$

$$= (c^4 - 6c^2 + 1) + i(4c^3 - 4c)$$

(b) $z = x+iy \Rightarrow z^2 = a+ib$

But $z^2 = (x+iy)^2 = (x+iy)(x+iy)$

$$= (x^2-y^2) + 2xyi$$

$$= a + ib$$

So Re: $x^2 - y^2 = a$ ①
Im: $2xy = b$ ②

Now, looking at the answer let us combine ① & ② as follows:

$$(x^2-y^2)^2 + (2xy)^2 = a^2 + b^2$$

(using the answer to guide you is NOT cheating!)

So $x^4 - 2x^2y^2 + y^4 + 4x^2y^2 = a^2 + b^2$

$\Rightarrow \quad x^4 + 2x^2y^2 + y^4 = a^2 + b^2$

$\therefore (x^2 + y^2)^2 = a^2 + b^2$

$\therefore \quad x^2 + y^2 = \pm\sqrt{a^2 + b^2}$

Now add (1) to Left & Right hand side

$$x^2 + y^2 + x^2 - y^2 = \pm\sqrt{a^2+b^2} + a$$

So $\quad 2x^2 = \pm\sqrt{a^2+b^2} + a$

[we could also have subtracted (1) to get another Equation:

$$x^2 + y^2 - (x^2 - y^2) = \pm\sqrt{a^2+b^2} - a$$

$\therefore \quad 2y^2 = \pm\sqrt{a^2+b^2} - a$]

[Another Equation can be derived this way:

$$2x^2 - 2y^2 = 2a \quad \text{and} \quad y = \frac{b}{2x}$$

So $\quad 2x^2 = 2a - 2y^2 = 2a - \dfrac{b^2}{4x^2}$

So $\quad 4x^4 = 4ax^2 - b^2$

$\Rightarrow \quad 4x^4 - 4ax^2 + b^2 = 0$

Completing the Square: $(2x^2 - 2a)^2 - 4a + b^2 = 0$

So $2x^2 - 2a = \pm\sqrt{b^2 - 4a^2}$

$$2x^2 = 2a \pm \sqrt{b^2 - 4a} \quad . \quad]$$

$z^4 + 6z^2 + 25 = 0$ can be written as

$(z^2)^2 + 6(z^2) + 25 = 0$ i.e. a quadratic in z^2

$$\therefore \quad z^2 = \frac{-6 \pm \sqrt{36 - 100}}{2}$$

$$= \frac{-6 \pm 8i}{2} = -3 \pm 4i$$

So $-3 \pm 4i = (a + ib)^2$
$= a^2 - b^2 + 2abi$

$\therefore -3 = a^2 - b^2$ & $\pm 4 = 2ab$
(1) (2)

By (2) we have $b = \pm \frac{2}{a}$ (3)

$\therefore$ into (1) : $-3 = a^2 - \left(\pm \frac{2}{a}\right)^2$

$$= a^2 - \frac{4}{a^2}$$

$\therefore \quad -3a^2 = a^4 - 4 \Rightarrow a^4 + 3a^2 - 4 = 0$
$\Rightarrow (a^2 + 4)(a^2 - 1) = 0$

a is Real so $a^2 + 4 = 0$ Not Valid

$\therefore \quad a^2 - 1 = 0 \Rightarrow a = -1, +1$

$a = 1 \Rightarrow b = \pm 2 \longrightarrow 1 + 2i,\ 1 - 2i$

$a = -1 \Rightarrow b = \mp 2 \longrightarrow -1 - 2i,\ -1 + 2i$